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# Duality in the quantum harmonic oscillator

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**Abstract**

Resuming the theme of a previous paper (Szafraniec F H 2001 *Math. Nachtr.* at press) we show that any abstract creation (or annihilation) operator is in *duality* with the *finite difference* one acting in $\ell^2$. This duality relation is generic and its spatial character allows both agents to be identified within the class of weighted shifts.

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**1**. Let $\mathcal{H}$ be a separable (but infinite-dimensional) complex Hilbert space. Given an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ and a sequence $\{\sigma_n\}_{n=0}^{\infty} \subset (0, +\infty)$ of *weights*, we say that an operator $S$ is a *weighted shift* (with respect to $\{e_n\}_{n=0}^{\infty}$ and with the weights $\{\sigma_n\}_{n=0}^{\infty}$) if $Se_n = \sigma_n e_{n+1}$, for $n = 0, 1, \ldots$ and $\text{lin}\{e_n\}_{n=0}^{\infty}$ is a core of $S$ (the latter means $\overline{S|_{\{e_n\}_{n=0}^{\infty}}} = \bar{S}$, with the 'bar' standing for the closure; since the left-hand side is always an operator, $S$ must necessarily be closable). Likewise, $T$ is called a *backward weighted shift* if $Te_n = \sigma_{n-1}e_{n-1}$, for $n = 0, 1, \ldots$ and $\text{lin}\{e_n\}_{n=0}^{\infty}$ is a core of $T$. Now we can say $S$ is a *creation operator* with respect to $\{e_n\}_{n=0}^{\infty}$ if it is a weighted shift with respect to $\{e_n\}_{n=0}^{\infty}$ with the weight sequence $\{\sqrt{n+1}\}_{n=0}^{\infty}$ and $T$ is an *annihilation operator* with respect to $\{e_n\}_{n=0}^{\infty}$ if it is a backward weighted shift with respect to $\{e_n\}_{n=0}^{\infty}$ with the same weight sequence $\{\sqrt{n+1}\}_{n=0}^{\infty}$. Thus, more explicitly, they act as

$$Se_n = \sqrt{n+1}\, e_{n+1} \qquad Te_n = \sqrt{n}\, e_{n-1} \qquad n = 0, 1, \ldots.$$

It is clear that with $S$ being a weighted shift and $T$ being a backward weighted shift, both with respect to the same basis and with the same weight sequence, $\text{lin}\{e_n\}_{n=0}^{\infty}$ is their core and

$$T \subset S^*.$$

It is because of this that we restrict ourselves exclusively to the creation operator, from which the annihilation operator can inherit its properties. Additionally, the two other members of the family, the number operator and the Hamiltonian, can also be derived from the creation operator. This may sound like a blasphemy to physicists but is a result, in fact, of the physicists having generously left all the details to the mathematicians; this is a matter of course, fortunately. Enjoying the benefits of this fact (instead of working again with formal relations so as to obtain, sooner or later, their representations), and knowing already that everything happens within a Hilbert space environment, we can try to build up an operator theory around the quantum harmonic oscillator (this is done in [10] as a result of the antecedent results quoted therein).

**2**. The *models* are, of course, of great importance, some of which are detailed below.

(A) Let $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. Consider the Hermite functions

$$h_n = 2^{-n/2}(n!)^{-1/2}\pi^{-1/4}\mathrm{e}^{-x^2/2}H_n$$

with $H_n$, the $n$-Hermite polynomial, defined as

$$H_n(x) = (-1)^n\mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2}.$$

Then

$$S = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right) \qquad T = \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

with $\mathcal{D}(S) = \mathcal{D}(T) \stackrel{\mathrm{df}}{=} \operatorname{lin}\{h_n;\ n = 0, 1, \ldots\}$ being the creation and annihilation operators, respectively. This is the *very classical* model.

(B) Now $\mathcal{H} = \mathcal{A}^2(\mathbb{C}, \pi^{-1}\exp(-|z|^2)\,\mathrm{d}x\,\mathrm{d}y)$, $S =$ 'multiplication by $Z$', $\mathcal{D}(S) = \mathbb{C}[Z]$. This is the *Segal–Bargmann* model [1], which is also classical.

(C) Here is a *discrete* model. Introduce in $\mathcal{H} = \ell^2$ a basis composed of Charlier sequences as follows. First the Charlier polynomials, $\{C_n^{(a)}\}_{n=0}^{\infty}$, $a > 0$, are determined by

$$\mathrm{e}^{-az}(1+z)^x = \sum_{n=0}^{\infty} C_n^{(a)}(x)\frac{z^n}{n!}.$$

They are orthogonal with respect to a nonnegative integer-supported measure according to

$$\sum_{x=0}^{\infty} C_m^{(a)}(x)C_n^{(a)}(x)\frac{\mathrm{e}^{-a}a^x}{x!} = \delta_{mn}a^n n! \qquad m, n = 0, 1, \ldots.$$

Define the *Charlier sequences* $c_n^{(a)}$, $n = 0, 1, \ldots$, as

$$c_n^{(a)}(x) = a^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}C_n^{(a)}(x)\mathrm{e}^{-\frac{a}{2}}a^{\frac{x}{2}}(x!)^{-\frac{1}{2}} \qquad x = 0, 1, \ldots.$$

The operators $S_a$ and $T_a$ defined as

$$(S_a f)(x) = \begin{cases} \sqrt{x}\, f(x-1) - \sqrt{a} f(x) & x = 1, 2, \ldots \\ -\sqrt{a}\, f(x) & x = 0 \end{cases}$$

$$(T_a f)(x) = \sqrt{x+1}\, f(x+1) - \sqrt{a}\, f(x) \qquad x = 0, 1, \ldots$$

for $f \in \mathcal{D}(S_a) = \mathcal{D}(T_a) \stackrel{\mathrm{df}}{=} \operatorname{lin}\{c_n^{(a)};\ n = 0, 1, \ldots\}$ (where $c_n^{(a)}$ are Charlier sequences) are the creation and annihilation ones, respectively (see [6] for details regarding this model). An inductive-limit procedure can be implemented (cf [9]) so as to approach (A) using the present model.

(D) Let $0 < A < 1$. Now $\mathcal{H}$ is the Hilbert space $\mathcal{X}^{(A)}$ of entire functions $f$ such that

$$\int_{\mathbb{R}^2} |f(x+\mathrm{i}y)|^2 \exp\left[Ax^2 - \frac{1}{A}y^2\right] \mathrm{d}x\,\mathrm{d}y < \infty.$$

Defining

$$h_n^{(A)}(z) = b_n(A)^{-1/2}\mathrm{e}^{-z^2/2}H_n(z) \qquad z \in \mathbb{C}$$

where $H_n$ is a Hermite polynomial which is now a complex variable, we get, by the orthogonality relation of [2], an orthonormal basis in $\mathcal{X}^{(A)}$. The operators $S^{(A)}$ and $T^{(A)}$ defined as (cf [7])

$$S^{(A)}f(z) = \sqrt{\frac{1-A}{2(1+A)}}\,[zf(z) - f'(z)]$$

$$T^{(A)}f(z) = \sqrt{\frac{1+A}{2(1-A)}}\,[zf(z) + f'(z)]$$

$$z \in \mathbb{C} \qquad f \in \mathrm{lin}\{h_n^{(A)}\}_{n=0}^{\infty}$$

are the creation and annihilation operators with respect to the basis $\{h_n^{(A)}\}_{n=0}^{\infty}$. An interesting feature of this model is that it provides a kind of *interpolating scale* between the very classical model (A), when $A \to 0+$, and the Segal–Bargmann model (B), here $A \to 1-$ (see [7] for details). Notice that, unlike in model (B), the measure involved in this case is *not* rotationally invariant.

**3**. Since the sequence $\{c_n^{(a)}\}_{n=0}^{\infty}$ of the Charlier sequences satisfies the relation

$$(-1)^x c_x^{(a)}(n) = (-1)^n c_n^{(a)}(x) \qquad x, n = 0, 1, \ldots \tag{1}$$

one can construct a family according to [8] (and also [11], proposition 2) of new bases in a Hilbert space starting from the original space as follows below.

*Fact A.* *Let $\{e_n\}_{n=0}^{\infty}$ be a basis in a separable Hilbert space $\mathcal{H}$. Then*

$$e_n^{(a)} \stackrel{\mathrm{df}}{=} \sum_{k=0}^{\infty} c_k^{(a)}(n) e_k \qquad n = 0, 1, \ldots \tag{2}$$

*defines a basis $\{e_n^{(a)}\}_{n=0}^{\infty}$ in $\mathcal{H}$. Moreover, $\{e_n\}_{n=0}^{\infty}$ can be recaptured from $\{e_n^{(a)}\}_{n=0}^{\infty}$ by the formula*

$$e_n = \sum_{k=0}^{\infty} c_n^{(a)}(k) e_k^{(a)} \qquad n = 0, 1, \ldots.$$

The creation operator satisfies the (formal) commutation relation

$$S^*S - SS^* = I \tag{3}$$

although the way back is *not* automatic (even if the precise meaning of (3) is clear); for this we refer to [10] where, among other things, *subnormality* gets involved. The following[1], however, is a rather trivial observation.

*Fact B.* *If $S$ satisfies (3), then so does $S - \lambda$ for any $\lambda \in \mathbb{C}$.*
However, it may be that neither $S$ nor $S - \lambda$ is a creation operator. The subsequent result ([4], and for its Weyl form see [5]) is related to this question.

[1] Some physically oriented consequences of this can be found in [3].

*Fact C.* *Let $S$ be a closed operator. If $S$ as well as $S - \lambda$ are weighted shifts with some $\lambda \in \mathbb{C} \setminus \{0\}$, then they are both multiplies of the creation operator.*

A version of this which makes the bases involved transparent is in [11].

*Fact D.* *Let $S$ be a closed operator.*

*(a) If $S$ is a creation operator with respect to $\{e_n\}_{n=0}^{\infty}$, then for any $a > 0$ the operator $S + \sqrt{a}\, I$ is a creation operator with respect to $\{e_n^{(a)}\}_{n=0}^{\infty}$ defined by (2).*

*Conversely,*

*(b) if $S$ is a weighted shift with respect to a basis $\{e_n\}_{n=0}^{\infty}$ and for some $a > 0$ the operator $S + \sqrt{a}\, I$ is a weighted shift with respect to some basis $\{f_n^{(a)}\}_{n=0}^{\infty}$, then $S$ is a creation operator with respect to $\{e_n\}_{n=0}^{\infty}$ and $f_n = e_n$ as well as $f_n^{(a)} = e_n^{(a)}$, $n = 0, 1, \ldots$, where the $e_n^{(a)}$ values are given as in (2) (accordingly $S + \sqrt{a}\, I$ is a creation operator with respect to $\{e_n^{(a)}\}_{n=0}^{\infty}$ for any $a > 0$).*

**4**. Let $S_a$ be the creation operator for model (C). This means that

$$\sqrt{n+1}\, c_{n+1}^{(a)}(x) = \begin{cases} \sqrt{x}\, c_n^{(a)}(x-1) - \sqrt{a}\, c_n^{(a)}(x) & x \geqslant 1 \\ -\sqrt{a}\, c_n^{(a)}(x) & x = 0 \end{cases} \tag{4}$$

and, by (1)

$$\sqrt{n+1}\, c_x^{(a)}(n+1) = \begin{cases} \sqrt{x}\, c_{x-1}^{(a)}(n) + \sqrt{a}\, c_x^{(a)}(n) & x \geqslant 1 \\ \sqrt{a}\, c_x^{(a)}(n) & x = 0. \end{cases} \tag{5}$$

The twin relation coming from the annihilation operator is

$$\sqrt{n}\, c_x^{(a)}(n-1) = \sqrt{x+1}\, c_{x+1}^{(a)}(n) + \sqrt{a}\, c_x n.$$

We can complete the above with the following theorem.

**Theorem 1.** *Let $S$ be a closed creation operator with respect to $\{e_n\}_{n=0}^{\infty}$ and let $S_a$ be the creation operator defined in model (C). Then*[2] *for any $a > 0$*

$$\begin{aligned} (-1)^n (S + \sqrt{a}\, I) e_n^{(a)} &= \sum_{k=0}^{\infty} (-1)^{k+1} (S_a c_n^{(a)})(k) e_k \\ (-1)^n (S^* + \sqrt{a}\, I) e_n^{(a)} &= \sum_{k=0}^{\infty} (-1)^{k+1} (S_a^* c_n^{(a)})(k) e_k \end{aligned} \qquad n = 0, 1, \ldots \tag{6}$$

*where the $e_n^{(a)}$ values are defined by (2).*

Relation (6) can be viewed as just a kind of *duality* between an arbitrary creation (annihilation, respectively) operator and a discrete one.

**Proof.** Take $N \in \mathbb{N}$. Using the fact that both $S$ and $S_a$ are creation operators, due to (5), we have

$$\begin{aligned} (S + \sqrt{a}\, I) \sum_{k=0}^{N} c_k^{(a)}(n) e_k &= \sum_{k=0}^{N} c_k^{(a)}(n) (S + \sqrt{a}\, I) e_k \\ &= \sum_{k=0}^{N} c_k^{(a)}(n) \sqrt{k+1}\, e_{k+1} + \sqrt{a} \sum_{k=0}^{N} c_k^{(a)}(n) e_k \end{aligned}$$

[2] If $\xi \in \ell^2$ we write $\xi(n)$ for its $n$-coordinate.

$$= \sum_{k=1}^{N} c_{k-1}^{(a)}(n)\sqrt{k}\, e_k + \sqrt{a} \sum_{k=0}^{N} c_k^{(a)}(n) e_k + c_N^{(a)} \sqrt{N+1}\, e_{N+1}$$
$$= \sum_{k=0}^{N} c_k^{(a)}(n+1)\sqrt{n+1}\, e_k + c_N^{(a)} \sqrt{N+1}\, e_{N+1}$$
$$= \sum_{k=0}^{N} (-1)^{n-k-1} c_{n+1}^{(a)}(k)\sqrt{n+1}\, e_k + c_N^{(a)} \sqrt{N+1}\, e_{N+1}$$
$$= (-1)^n \sum_{k=0}^{\infty} (-1)^{k+1} (S_a c_n^{(a)})(k) e_k + c_N^{(a)} \sqrt{N+1}\, e_{N+1}.$$

Since $S$ is closed and $c_N^{(a)}\sqrt{N+1} \to 0$ as $N \to \infty$ (this is due to $\{c_N^{(a)}\sqrt{N+1}\}_{N=0}^{\infty} \in \ell^2$ being in the range of $S_a$) every $e_n^{(a)}$ is in the domain of $S$ and, consequently, the first of (6) holds. The other can be deduced in the same way using (4) in place of (5). □

The point is that we have a kind of converse to the above; actually there are a couple of versions which we itemize as separate results. In all of them we *assume* that

$$S \text{ is closed and } \{e_n\}_{n=0}^{\infty}, \{e_n^{(a)}\}_{n=0}^{\infty} \subset \mathcal{D}(S).$$

**Theorem 2.** *Suppose $S$ is a weighted shift with respect to $\{e_n\}_{n=0}^{\infty}$ and $S_\square$ is a weighted shift with respect to $\{c_n^{(a)}\}_{n=0}^{\infty}$. If for some $a > 0$ either*

$$(-1)^n (S + \sqrt{a}\, I) e_n^{(a)} = \sum_{k=0}^{\infty} (-1)^{k+1} (S_\square\, c_n^{(a)})(k) e_k \qquad n = 0, 1, \ldots$$

*or*

$$(-1)^n (S^* + \sqrt{a}\, I) e_n^{(a)} = \sum_{k=0}^{\infty} (-1)^{k+1} (S_\square^*\, c_n^{(a)})(k) e_k \qquad n = 0, 1, \ldots$$

*then $S$ is a creation operator with respect to $\{e_n\}_{n=0}^{\infty}$ and $S_\square$ is a creation operator with respect to $\{c_n^{(a)}\}_{n=0}^{\infty}$ (that is, $\bar{S}_\square = \bar{S}_a$).*

**Proof.** Suppose the first of the duality relations hold. Let $\{\sigma_n\}_{n=0}^{\infty}$ be the weight sequence of $S$ and $\{\tau_n\}_{n=0}^{\infty}$ is that of $S_\square$. Then by (1) we have

$$\tau_n \sum_{k=0}^{\infty} c_k^{(a)}(n+1) e_k = \sum_{k=0}^{\infty} (-1)^{n+1-k} \tau_n c_{n+1}^{(a)}(k) e_k = \sum_{k=0}^{\infty} (-1)^{n+1-k} (S_\square\, c_n^{(a)})(k) e_k$$
$$= (S + \sqrt{a}\, I) \sum_{k=0}^{\infty} c_k^{(a)}(n) e_k = \sum_{k=0}^{\infty} (\sigma_k c_k^{(a)}(n) e_{k+1} + \sqrt{a}\, c_k^{(a)}(n) e_k)$$
$$= \sum_{k=1}^{\infty} \sigma_{k-1} c_{k-1}^{(a)}(n) e_k + \sqrt{a} \sum_{k=0}^{\infty} c_k^{(a)}(n) e_k.$$

Equating the coefficients of $e_k$ we get

$$\tau_n c_k^{(a)}(n+1) = \begin{cases} \sigma_{k-1} c_{k-1}^{(a)}(n) + \sqrt{a}\, c_k^{(a)}(n) & k = 1, 2, \ldots \\ \sqrt{a}\, c_k(n) & k = 0 \end{cases}$$

and comparing this with (5) enables us to get the correct result. □

**Theorem 3.** *Suppose $S$ is a weighted shift with respect to some basis $\{f_n\}_{n=0}^{\infty}$ with $f_0 = e_0$. If for some $a > 0$ either*

$$(-1)^n(S + \sqrt{a}\, I)e_n^{(a)} = \sum_{k=0}^{\infty}(-1)^{k+1}(S_a c_n^{(a)})(k)e_k \qquad n = 0, 1, \ldots$$

*or*

$$(-1)^n(S^* + \sqrt{a}\, I)e_n^{(a)} = \sum_{k=0}^{\infty}(-1)^{k+1}(S_a^* c_n^{(a)})(k)e_k \qquad n = 0, 1, \ldots$$

*where $S_a$ is the creation operator of (C), then $f_n = e_n$, $n = 0, 1, \ldots$ and $S$ is the creation operator.*

**Proof.** Now consider the first possibility; insert into it $e_k = \sum_{s=0}^{\infty} \xi_s^{(k)} f_k$ and perform the shifting. What we get, via (1) and (5), is

$$\sum_{s=1}^{\infty}\sum_{k=0}^{\infty} c_k^{(a)}(n)(\sigma_s \xi_{s-1}^{(k)} + \sqrt{a}\, \xi_s^{(k)}) f_s + \sqrt{a} \sum_{k=0}^{\infty} c_k^{(a)}(n)\xi_0^{(k)} f_0 = \sum_{s=0}^{\infty}\left(\sum_{k=o}^{\infty} \sqrt{k+1}\, c_k^{(a)}(n)\xi_s^{(k)}\right) f_s$$

where $\{\sigma_n\}_{n=0}^{\infty}$ is the sequence of weights for $S$. Equating coefficients at $f_s$ gives

$$\sqrt{k+1}\, \xi_s^{(k+1)} = \begin{cases} 0 & s = 0 \\ \sigma_s \xi_{s-1}^{(k)} & s = 1, 2, \ldots. \end{cases}$$

Since $\xi_k^{(0)} = \delta_{k,0}$ for $k = 0, 1, \ldots$ ($f_0 = e_0$) we get $\xi_s^{(k)} = \sigma_k' \delta_{k,s}$, which means that $S$ is a weighted shift with respect to $\{e_n\}_{n=0}^{\infty}$. This brings us to theorem 1. □

**Theorem 4.** *Suppose $S_\square$ is a weighted shift in $\ell^2$ with respect to some basis $\{d_n\}_{n=0}^{\infty}$. If $S$ is a creation operator with respect to $\{e_n\}_{n=0}^{\infty}$ as well as for some $a > 0$ $d_0 = c_0^{(a)}$ and either*

$$(-1)^n(S + \sqrt{a}\, I)e_n^{(a)} = \sum_{k=0}^{\infty}(-1)^{k+1}(S_a c_n^{(a)})(k)e_k \qquad n = 0, 1, \ldots$$

*or*

$$(-1)^n(S^* + \sqrt{a}\, I)e_n^{(a)} = \sum_{k=0}^{\infty}(-1)^{k+1}(S_a^* c_n^{(a)})(k)e_k \qquad n = 0, 1, \ldots$$

*then $d_n = c_n^{(a)}$, $n = 0, 1, \ldots$ and $S_\square$ is the creation operator of model (C); that is, $\bar{S_\square} = \bar{S}_a$.*

The proof uses the same sort of argument as that of 2.

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